

An exactly solvable deformation of the Coulomb problem associated with the Taub-NUT metric

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Abstract

In this paper we quantize the N -dimensional classical Hamiltonian system

$$\mathcal{H} = \frac{|\mathbf{q}|}{2(\eta + |\mathbf{q}|)} \mathbf{p}^2 - \frac{k}{\eta + |\mathbf{q}|},$$

that can be regarded as a deformation of the Coulomb problem with coupling constant k , that it is smoothly recovered in the limit $\eta \rightarrow 0$. Moreover, the kinetic energy term in \mathcal{H} is just the one corresponding to an N -dimensional Taub-NUT space, a fact that makes this system relevant from a geometric viewpoint. Since the Hamiltonian \mathcal{H} is known to be maximally superintegrable, we propose a quantization prescription that preserves such superintegrability in the quantum mechanical setting. We show that, to this end, one must choose as the kinetic part of the Hamiltonian the conformal Laplacian of the underlying Riemannian manifold, which combines the usual Laplace–Beltrami operator on the Taub–NUT manifold and a multiple of its scalar curvature. As a consequence, we obtain a novel exactly solvable deformation of the quantum Coulomb problem, whose spectrum is computed in closed form for positive values of η and k , and showing that the well-known maximal degeneracy of the flat system is preserved in the deformed case. Several interesting algebraic and physical features of this new exactly solvable quantum system are analysed, and the quantization problem for negative values of η and/or k is also sketched.

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1 Introduction

Let us consider the two-parameter family of N -dimensional (ND) classical Hamiltonian systems given by

$$\mathcal{H} = \mathcal{T}(\mathbf{q}, \mathbf{p}) + \mathcal{U}(\mathbf{q}) = \frac{|\mathbf{q}|}{2(\eta + |\mathbf{q}|)} \mathbf{p}^2 - \frac{k}{\eta + |\mathbf{q}|}, \quad (1.1)$$

where η and k are real parameters, $\mathbf{q}, \mathbf{p} \in \mathbb{R}^N$ are conjugate coordinates and momenta with canonical Poisson bracket $\{q_i, p_j\} = \delta_{ij}$ and

$$\mathbf{q}^2 = \sum_{i=1}^N q_i^2, \quad \mathbf{p}^2 = \sum_{i=1}^N p_i^2, \quad |\mathbf{q}| = \sqrt{\mathbf{q}^2}.$$

Clearly, the full Hamiltonian (1.1) can be regarded as an η -deformation of the ND Euclidean Coulomb problem with coupling constant k , since the limit $\eta \rightarrow 0$ yields

$$\mathcal{H} = \frac{1}{2} \mathbf{p}^2 - \frac{k}{|\mathbf{q}|}.$$

Contemporarily, the system (1.1) can also be interpreted as a Hamiltonian defined on a curved space, since the kinetic energy term $\mathcal{T}(\mathbf{q}, \mathbf{p})$ provides the geodesic motion on the underlying ND curved manifold $\mathcal{M} = \mathbb{R}^N \setminus \{\mathbf{0}\}$ with metric

$$ds^2 = \left(1 + \frac{\eta}{|\mathbf{q}|}\right) d\mathbf{q}^2, \quad |\mathbf{q}| \neq 0, \quad (1.2)$$

and scalar curvature given by

$$R = \eta(N-1) \frac{4(N-3)r + 3\eta(N-2)}{4r(\eta+r)^3}, \quad (1.3)$$

where we have introduced the radial coordinate $r = |\mathbf{q}|$. Note that the limit $\eta \rightarrow 0$ provides the flat/Euclidean expressions $ds^2 = d\mathbf{q}^2$ and $R = 0$. Therefore, we are dealing with a system defined on a conformally flat and spherically symmetric space \mathcal{M} with metric

$$ds^2 = f(r)^2 d\mathbf{q}^2, \quad (1.4)$$

whose conformal factor reads

$$f(r) = \sqrt{1 + \frac{\eta}{r}}. \quad (1.5)$$

It turns out that the mathematical and physical relevance of the Hamiltonian (1.1) relies on two important facts. On one hand, when $N = 3$ the Hamiltonian \mathcal{H} is directly related to a reduction [1] of the geodesic motion on the Taub-NUT space [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. On the other hand, it was shown in [13] that (1.1) defines a maximally superintegrable classical system, that is, \mathcal{H} is endowed with the maximum possible number of $(2N - 1)$ functionally independent integrals of motion, all of which are, in this case, quadratic in the momenta.

In this paper we will present a quantization of (1.1) that preserves the maximal superintegrability of the system. This result will provide an exactly solvable deformation of the

Coulomb problem, whose eigenvalue problem will be computed in detail with an emphasis on the cases $\eta > 0$ and $k > 0$. As expected, the spectrum of the standard ND Coulomb problem will be recovered in the limit $\eta \rightarrow 0$.

It is worth stressing that the quantization of the Hamiltonian (1.1) is by no means straightforward, since the kinetic energy term $\mathcal{T}(\mathbf{q}, \mathbf{p})$ generates an ordering ambiguity when the classical position and momenta are replaced by the corresponding operators. We shall see how this problem can be solved by following the quantization procedure proposed in [14, 15] for another maximally superintegrable quantum system on a different ND curved space: the so-called Darboux III oscillator system, which is an exactly solvable deformation of the harmonic oscillator potential that is associated to the Darboux III space [16, 17]. Therefore, new exactly solvable deformations of the oscillator and Coulomb problems can be obtained when certain curved spaces with prescribed integrability properties are considered. Moreover, we remark that the existence of additional integrals of the motion associated with the maximal superintegrability of (1.1), gives rise to an $\mathfrak{so}(N+1)$ Lie symmetry algebra identical to the one underlying the ND Euclidean Coulomb system. We will also show that this fact makes it possible to compute formally the discrete spectrum of the system in a very efficient way.

The structure of the paper is as follows. In the next section, the maximal superintegrability of the classical Hamiltonian \mathcal{H} (1.1) is revisited (see [13, 18]). In particular, the geometric and dynamical features of the space (1.2) are studied as well as the connection between \mathcal{H} and the Taub-NUT metrics. In Section 3 we present a quantization for \mathcal{H} that preserves the full symmetry algebra of the classical system and, therefore, its maximal superintegrability. Explicitly, we shall prove that this is achieved through the *conformal Laplacian quantization* [15], namely

$$\hat{\mathcal{H}}_c = -\frac{\hbar^2}{2}\Delta_c + \mathcal{U} = -\frac{\hbar^2}{2}\left(\Delta_{LB} - \frac{(N-2)}{4(N-1)}R\right) + \mathcal{U}, \quad (1.6)$$

where R is the scalar curvature, here given by (1.3), and Δ_c is the conformal Laplacian [19]. Notice that the conformal Laplacian Δ_c is the sum of the usual Laplace–Beltrami operator Δ_{LB} on the curved manifold \mathcal{M} plus a multiple of the scalar curvature R of the manifold, while \mathcal{U} is just the classical potential given in (1.1) (see, e.g., the comprehensive reference [20] and [21, 22, 23]). In order to prove this result we shall make use of the fact that (1.6) can be related through a similarity transformation to the Hamiltonian obtained by means of the so-called *direct Schrödinger quantization* prescription [15, 18], namely

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2f(r)^2}\Delta + \mathcal{U}, \quad (1.7)$$

where $f(r) = f(|\mathbf{q}|)$ is the conformal factor of the metric (1.4) and Δ is the Laplacian in the \mathbf{q} coordinates. The eigenvalue problem for these Hamiltonians will be rigorously solved in Section 4, where it is found that, for positive k and η , the discrete spectrum of the system is a smooth deformation of the ND Euclidean Coulomb problem spectrum in terms of the parameter η and, as expected, the quantum system presents the same maximal degeneracy as the ND hydrogen atom. The eigenvalue problem for other possible values of k and η are also sketched, and the paper concludes with some remarks and open problems.

2 The classical system

The maximal superintegrability of the classical system \mathcal{H} is explicitly stated through the following result [13], that can be readily proven through direct computations.

Proposition 1. (i) *The Hamiltonian \mathcal{H} (1.1) is endowed with $(2N - 3)$ angular momentum integrals given by $(m = 2, \dots, N)$*

$$C^{(m)} = \sum_{1 \leq i < j \leq m} (q_i p_j - q_j p_i)^2, \quad C_{(m)} = \sum_{N-m < i < j \leq N} (q_i p_j - q_j p_i)^2, \quad C^{(N)} = C_{(N)} \equiv \mathbf{L}^2, \quad (2.1)$$

where \mathbf{L}^2 is the square of the total angular momentum.

(ii) *The Hamiltonian \mathcal{H} Poisson-commutes with the \mathcal{R}_i components $(i = 1, \dots, N)$ of the Runge–Lenz N -vector given by*

$$\mathcal{R}_i = \sum_{j=1}^N p_j (q_j p_i - q_i p_j) + \frac{q_i}{|\mathbf{q}|} (\eta \mathcal{H} + k).$$

(iii) *The set $\{\mathcal{H}, C^{(m)}, C_{(m)}, \mathcal{R}_i\}$, with $m = 2, \dots, N$ and a fixed index i , is formed by $(2N - 1)$ functionally independent functions.*

Note that the following functional relation between the Runge–Lenz vector \mathbf{R} , the angular momentum \mathbf{L} and the Hamiltonian \mathcal{H} holds:

$$\mathbf{R}^2 = \sum_{i=1}^N \mathcal{R}_i^2 = 2\mathbf{L}^2 \mathcal{H} + (\eta \mathcal{H} + k)^2. \quad (2.2)$$

Therefore, Proposition 1 establishes that \mathcal{H} is a maximally superintegrable Hamiltonian that is endowed with an $\mathfrak{so}(N)$ Lie–Poisson symmetry, since it is constructed on a spherically symmetric space. Explicitly, the functions $J_{ij} = q_i p_j - q_j p_i$ with $i < j$ and $i, j = 1, \dots, N$ span the $\mathfrak{so}(N)$ Lie–Poisson algebra

$$\{J_{ij}, J_{ik}\} = J_{jk}, \quad \{J_{ij}, J_{jk}\} = -J_{ik}, \quad \{J_{ik}, J_{jk}\} = J_{ij}, \quad i < j < k,$$

and the $(2N - 3)$ angular momentum integrals $C^{(m)}$ and $C_{(m)}$ (2.1) correspond to the quadratic Casimirs of some rotation subalgebras $\mathfrak{so}(m) \subset \mathfrak{so}(N)$.

Moreover, if we also take into account the additional integrals of the motion \mathcal{R}_i , then it can be checked that \mathcal{R}_i and J_{ij} span the Lie–Poisson algebra $\mathfrak{so}(N + 1)$. In fact, we immediately obtain that

$$\{J_{ij}, \mathcal{R}_k\} = \delta_{ik} \mathcal{R}_j - \delta_{jk} \mathcal{R}_i,$$

together with the quadratic Poisson bracket

$$\{\mathcal{R}_i, \mathcal{R}_j\} = -2\mathcal{H} J_{ij}.$$

Nevertheless, if we now define

$$\tilde{J}_{0i} = \frac{\mathcal{R}_i}{\sqrt{-2\mathcal{H}}}, \quad \tilde{J}_{ij} = J_{ij}, \quad (2.3)$$

we find that the functions \tilde{J}_{ij} close the Lie–Poisson algebra $\mathfrak{so}(N+1)$:

$$\{\tilde{J}_{ij}, \tilde{J}_{ik}\} = \tilde{J}_{jk}, \quad \{\tilde{J}_{ij}, \tilde{J}_{jk}\} = -\tilde{J}_{ik}, \quad \{\tilde{J}_{ik}, \tilde{J}_{jk}\} = \tilde{J}_{ij}, \quad i < j < k,$$

with $i, j, k = 0, 1, \dots, N$. Therefore, \mathcal{H} turns out to be expressible as a function of the quadratic Casimir function for $\mathfrak{so}(N+1)$, since from (2.2) we have

$$\sum_{i=1}^N \tilde{J}_{0i}^2 + \mathbf{L}^2 = -\frac{(\eta\mathcal{H} + k)^2}{2\mathcal{H}}.$$

The Hamiltonian \mathcal{H} can also be expressed in terms of hyperspherical coordinates r, θ_j , and canonical momenta p_r, p_{θ_j} , ($j = 1, \dots, N-1$) defined by

$$q_j = r \cos \theta_j \prod_{k=1}^{j-1} \sin \theta_k, \quad 1 \leq j < N, \quad q_N = r \prod_{k=1}^{N-1} \sin \theta_k, \quad (2.4)$$

so that

$$r = |\mathbf{q}|, \quad \mathbf{p}^2 = p_r^2 + r^{-2} \mathbf{L}^2, \quad \mathbf{L}^2 = \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2 \theta_k}. \quad (2.5)$$

Thus, for a given value of \mathbf{L}^2 , the Hamiltonian \mathcal{H} can be written as a 1D radial system:

$$\mathcal{H}(r, p_r) = \mathcal{T}(r, p_r) + \mathcal{U}(r) = \frac{r}{2(\eta + r)} \left(p_r^2 + \frac{\mathbf{L}^2}{r^2} \right) - \frac{k}{\eta + r}. \quad (2.6)$$

2.1 Geometric interpretation

We stress that all the above Poisson algebraic results hold for *any* value of the deformation parameter η and of the coupling constant k . Nevertheless, \mathcal{H} comprises *different* classes of physical systems. In particular, the domain of the variable r in \mathcal{M} depends on the sign of η :

$$\eta > 0 : \quad r \in (0, \infty); \quad \eta < 0 : \quad r \in (|\eta|, \infty). \quad (2.7)$$

In particular, the case with $\eta > 0$ turns out to be a system on a space:

- With *positive* (nonconstant) scalar curvature if $N \geq 3$, which is given by (1.3). Note that in this case the scalar curvature always diverges in the limit $r \rightarrow 0$, whereas it tends to zero for $r \rightarrow \infty$, as shown in Figure 1 for $N = 3$.
- With *negative* and finite (nonconstant) scalar curvature in the case $N = 2$, where (1.3) reads

$$R = -\frac{\eta}{(\eta + r)^3},$$

and whose $r \rightarrow 0$ limit is obviously $-1/\eta^2$ and whose $r \rightarrow \infty$ limit also vanishes. This curvature is plotted in Figure 2.

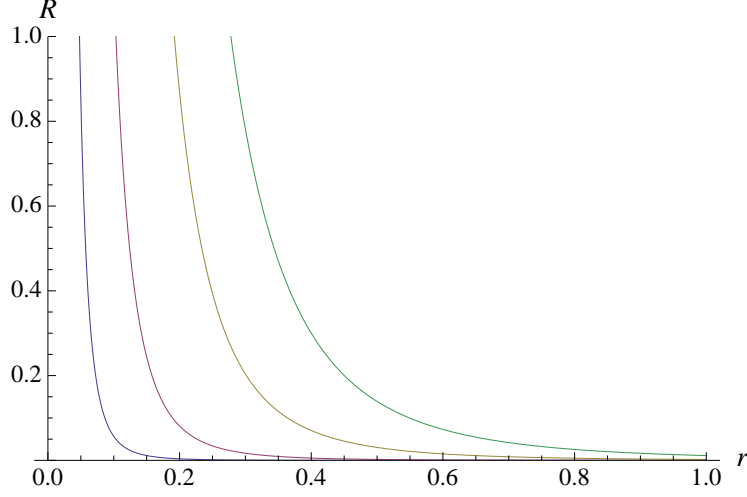


Figure 1: Scalar curvature (1.3) of the Taub-NUT space for $N = 3$ and where $\eta = \{0.002, 0.01, 0.04, 0.1\}$.

It is worth noticing that there is a codimension-1 Riemannian embedding of \mathcal{M} in Euclidean space. Specifically, let us denote by $d\omega^2$ the canonical metric on the sphere \mathbb{S}^{N-1} . The metric (1.2), which can be written as

$$ds^2 = \left(1 + \frac{\eta}{|\mathbf{q}|}\right) d\mathbf{q}^2 = \left(1 + \frac{\eta}{r}\right) (dr^2 + r^2 d\omega^2)$$

is then recovered from the Euclidean metric

$$dx_1^2 + \dots + dx_{N+1}^2$$

in \mathbb{R}^{N+1} upon setting

$$(x_1, \dots, x_N) = r \sqrt{1 + \frac{\eta}{r}} \omega, \quad x_{N+1} = z(r),$$

where $\omega = \omega(\theta_1, \dots, \theta_{N-1})$ parametrizes a point in \mathbb{S}^{N-1} via the hyperspherical coordinates used in (2.4), r takes values according to (2.7) and the function $z(r)$ is defined as

$$z(r) = \int_1^r \left(\frac{\eta(4r' + 3\eta)}{4r'(r' + \eta)} \right)^{\frac{1}{2}} dr'.$$

For $N = 2$, this embedding is represented in Figure 3. The fact that the surface is negatively curved and asymptotically flat is apparent from these pictures.

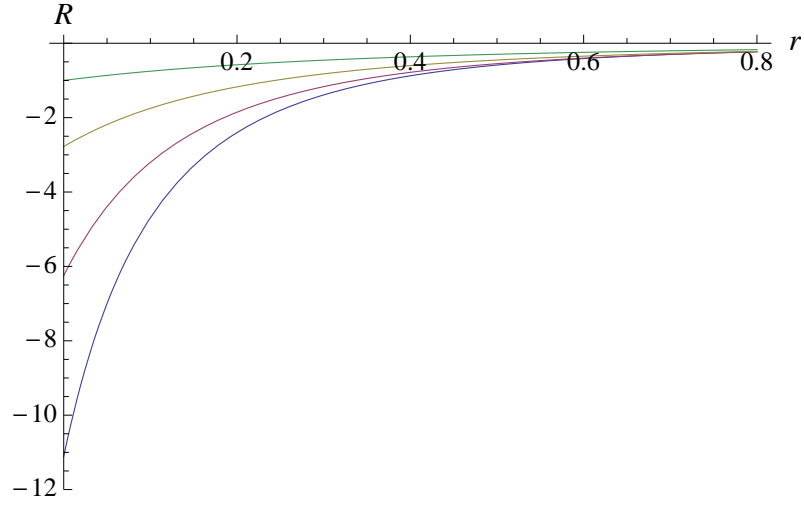


Figure 2: Curvature (1.3) of the Taub-NUT space for $N = 2$ and with $\eta = \{0.3, 0.4, 0.6, 1\}$.

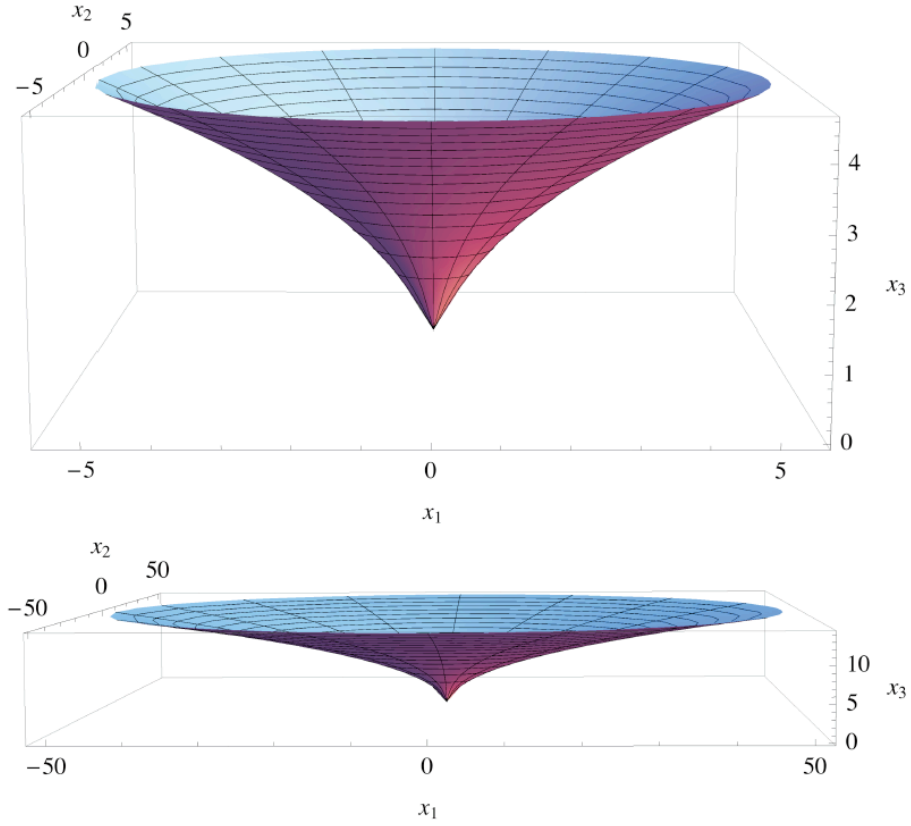


Figure 3: 3D Euclidean embedding of the Taub-NUT space for $N = 2$ with $\eta = 1$ plotted for $r \in [0, 5]$ and $r \in [0, 50]$.

2.2 The connection with the Taub-NUT system

It is important to stress that the case with $\eta > 0$ and $k < 0$ is the one related with the Taub-NUT system studied in [1], which describes the 3D reduction of the geodesic motion on the 4D Eucliden Taub-NUT metric. In particular, let us write the Taub-NUT system \mathcal{H}_T in the form [1, 24]:

$$\begin{aligned}\mathcal{H}_T &= \frac{\mathbf{p}^2}{2(1+4m/r)} + \frac{\mu^2}{2(4m)^2} \left(1 + \frac{4m}{r}\right) \\ &= \frac{r\mathbf{p}^2}{2(4m+r)} + \frac{\mu^2 r/(4m)^2}{2(4m+r)} + \frac{\mu^2/(4m)}{4m+r} + \frac{\mu^2}{2r(4m+r)}.\end{aligned}\quad (2.8)$$

Next, if we consider the Hamiltonian \mathcal{H} (1.1) with

$$\eta = 4m, \quad k = -\frac{\mu^2}{8m},$$

and we add a constant potential, then we get

$$\mathcal{H} + \frac{\mu^2}{2(4m)^2} = \frac{r\mathbf{p}^2}{2(4m+r)} + \frac{\mu^2 r/(4m)^2}{2(4m+r)} + \frac{\mu^2/(4m)}{4m+r} = \mathcal{H}_T - \frac{\mu^2}{2r(4m+r)},$$

thereby showing the equivalence between \mathcal{H} and the Taub-NUT system (2.8) up to an additive term which is just a centrifugal potential, that can be nevertheless reabsorbed by changing the value of the (conserved) total angular momentum of the system (i.e., replacing \mathbf{L}^2 by $\mathbf{L}^2 + \mu^2$ in (2.5)).

It is worth recalling that the system \mathcal{H} (1.1) is connected with the one introduced in [4] by considering the asymptotic motion of monopoles when its separation is much greater than their radii, which was already presented as a “non-trivial deformation of the Coulomb problem” in that reference. In what follows we will be mainly concerned with the case where k and η are both positive, whose spectral properties are particularly interesting, but we will make some remarks about the other choices of sign too.

2.3 Dynamical interpretation

We shall consider that $k > 0$, in order to be able to recover the hydrogen atom as the flat $\eta \rightarrow 0$ limit system, and $\eta > 0$ ensuring that $r \in (0, \infty)$. Note that if $k < 0$ the limiting case would be the repulsive Coulomb problem, for which there are no bounded trajectories (or normalizable eigenfunctions).

Unlike the standard Coulomb potential, the “deformed” one $\mathcal{U}(r)$ (2.6) is finite at $r = 0$, though both keep the same asymptotic behavior when $r \rightarrow \infty$, namely,

$$\mathcal{U}(r) = -\frac{k}{\eta+r}, \quad \mathcal{U}(0) = -\frac{k}{\eta}, \quad \lim_{r \rightarrow \infty} \mathcal{U}(r) = 0. \quad (2.9)$$

This potential is shown in Figure 4 for several values of η . In fact, the deformed Coulomb potential for a given r is just the flat Coulomb one for a “shifted” radial coordinate $r + \eta$.

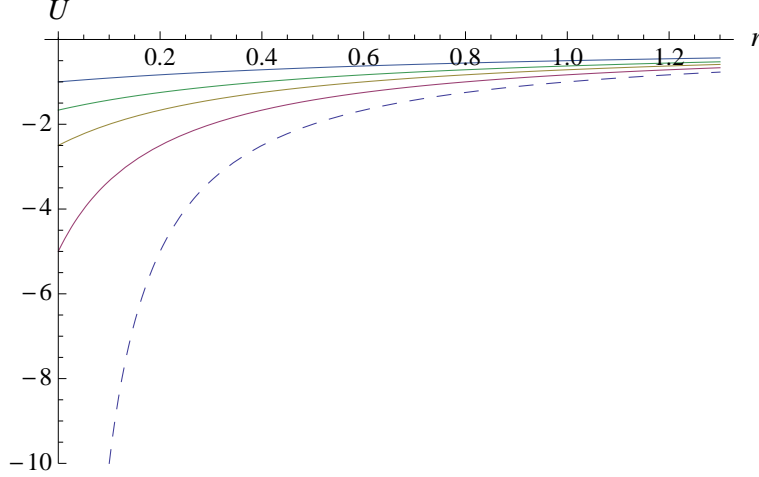


Figure 4: The deformed Coulomb potential (2.9) for $\eta = \{0, 0.2, 0.4, 0.6, 1\}$ with $k = 1$. The Euclidean Coulomb potential ($\eta = 0$) is represented by the dashed line.

However, since the underlying manifold \mathcal{M} is non-flat, the radial motion can be better understood by introducing a classical effective potential. In fact, let us consider the canonical transformation defined by

$$P(r, p_r) = \sqrt{\frac{r}{\eta + r}} p_r, \quad Q(r) = \sqrt{r(\eta + r)} + \eta \ln(\sqrt{r} + \sqrt{\eta + r}), \quad (2.10)$$

such that $\{Q, P\} = 1$ and $Q \in (\eta \ln \sqrt{\eta}, \infty)$. In this way, we obtain that the radial Hamiltonian (2.6) is transformed into

$$\mathcal{H}(Q, P) = \frac{1}{2} P^2 + \mathcal{U}_{\text{eff}}(Q), \quad \mathcal{U}_{\text{eff}}(Q(r)) = \frac{\mathbf{L}^2}{2r(\eta + r)} - \frac{k}{\eta + r}. \quad (2.11)$$

Consequently, the radial motion for the classical system can be described as the one of a particle on $r \in (0, \infty)$ under the action of the effective potential \mathcal{U}_{eff} . This potential is represented in Figure 5, where it can be appreciated that the effect of the η -deformation is to raise the minimum of \mathcal{U}_{eff} and to increase slightly the radius of the circular orbit for the system. Moreover, this classical effective potential does not depend on the dimension N of the system. On the other hand, if the angular momentum vanishes the effective potential is just $\mathcal{U}(r)$ (2.9), depicted in Figure 4, which is never singular at the origin whenever $\eta \neq 0$.

3 Maximally superintegrable quantization

Let us consider an ND curved space whose metric and associated classical kinetic term are given by

$$ds^2 = \sum_{i,j=1}^N g_{ij}(\mathbf{q}) dq_i dq_j, \quad \mathcal{T}(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i,j=1}^N g^{ij}(\mathbf{q}) p_i p_j.$$

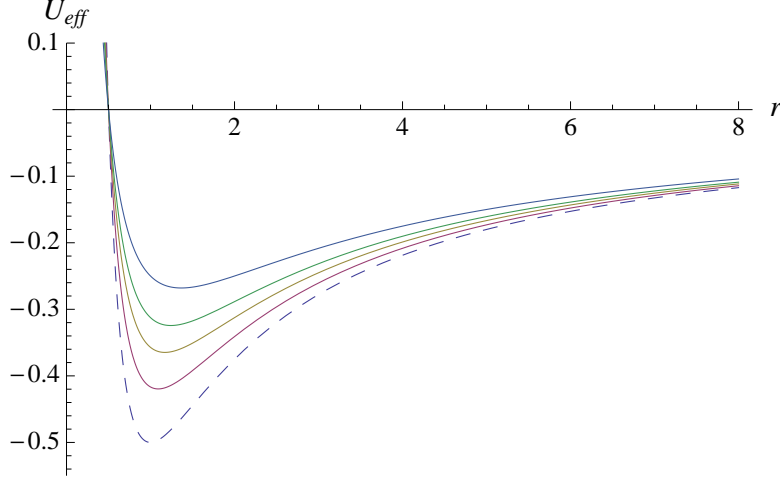


Figure 5: The radial effective potential (2.11) for $\eta = \{0, 0.2, 0.4, 0.6, 1\}$ with $\mathbf{L}^2 = 1$ and $k = 1$. The Euclidean Coulomb effective potential ($\eta = 0$) is represented by the dashed line.

Then the corresponding Laplace–Beltrami operator reads

$$\Delta_{\text{LB}} = \sum_{i,j=1}^N \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j, \quad (3.1)$$

where g^{ij} is the inverse of the metric tensor g_{ij} and g is the corresponding determinant. In principle, this operator could be used in order to define the quantum kinetic energy operator in the form (see, for instance, [17, 25])

$$\hat{\mathcal{T}}_{\text{LB}} = -\frac{\hbar^2}{2} \Delta_{\text{LB}}.$$

However, it turns out that a more popular quantization prescription used in the analysis of scalar field theories in General Relativity or when dealing with quantization on arbitrary Riemannian manifolds [20, 21, 22, 23] is given by the so-called *conformal Laplacian* (or Yamabe operator)

$$\Delta_{\text{c}} = \Delta_{\text{LB}} - \frac{(N-2)}{4(N-1)} R,$$

where R is the scalar curvature of the underlying manifold. Note that both prescriptions coincide when either $N = 2$ or $R = 0$. Moreover, it was proven in [15] for the Darboux III oscillator system, that this conformal Laplacian provides a maximally superintegrable quantization, since the quantum counterparts of all the constants of the motion of the classical system were explicitly found and, furthermore, the exact solvability of the system was obtained by making use of all these symmetries.

In what follows, we present a maximally superintegrable quantization for the deformed Coulomb system \mathcal{H} (1.1), and we will explicitly obtain the full set of $(2N-2)$ algebraically independent quantum observables that commute with the Hamiltonian. Hereafter, we will

use the standard definitions for the quantum position $\hat{\mathbf{q}}$ and momenta $\hat{\mathbf{p}}$ operators:

$$\hat{q}_i \psi(\mathbf{q}) = q_i \psi(\mathbf{q}), \quad \hat{p}_i \psi(\mathbf{q}) = -i\hbar \frac{\partial \psi(\mathbf{q})}{\partial q_i}, \quad [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad i, j = 1, \dots, N,$$

together with the conventions

$$\nabla = \left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_N} \right), \quad \Delta = \nabla^2 = \frac{\partial^2}{\partial^2 q_1} + \dots + \frac{\partial^2}{\partial^2 q_N}, \quad \mathbf{q} \cdot \nabla = \sum_{i=1}^N q_i \frac{\partial}{\partial q_i}.$$

Note that the operator $|\hat{\mathbf{q}}|$ is defined as $|\hat{\mathbf{q}}| \psi(\mathbf{q}) = |\mathbf{q}| \psi(\mathbf{q})$.

3.1 Conformal Laplacian quantization

It is straightforward to check that the Laplace–Beltrami operator (3.1) for the manifold \mathcal{M} with metric tensor given by (1.2) reads

$$\Delta_{\text{LB}} = \frac{|\mathbf{q}|}{\eta + |\mathbf{q}|} \Delta - \frac{\eta(N-2)}{2|\mathbf{q}|(|\mathbf{q}| + \eta)^2} (\mathbf{q} \cdot \nabla),$$

while the scalar curvature R of the Taub–NUT space is given by (1.3). With these two ingredients, a straightforward computation shows that the conformal Laplacian quantization of the classical deformed Coulomb system \mathcal{H} (1.1) would be given by the Hamiltonian operator (1.6), namely

$$\hat{\mathcal{H}}_c = -\frac{\hbar^2}{2} \Delta_{\text{LB}} + \hbar^2 \eta(N-2) \frac{4(N-3)|\mathbf{q}| + 3\eta(N-2)}{32|\mathbf{q}|(\eta + |\mathbf{q}|)^3} - \frac{k}{\eta + |\mathbf{q}|}. \quad (3.2)$$

In order to prove that $\hat{\mathcal{H}}_c$ is a maximally superintegrable quantization, the $(2N-2)$ algebraically independent operators commuting with $\hat{\mathcal{H}}_c$ have to be explicitly found. By following [15], this can be achieved by considering the Hamiltonian $\hat{\mathcal{H}}_c$ in the form

$$\hat{\mathcal{H}}_c = e^f \hat{\mathcal{H}} e^{-f}, \quad f(\mathbf{q}) = \frac{2-N}{4} \ln \left(1 + \frac{\eta}{|\mathbf{q}|} \right), \quad (3.3)$$

which is provided by a similarity transformation from the so-called “direct Schrödinger quantization” prescription (1.7) for the deformed Coulomb system:

$$\hat{\mathcal{H}} = \frac{-\hbar^2 |\mathbf{q}|}{2(\eta + |\mathbf{q}|)} \Delta - \frac{k}{\eta + |\mathbf{q}|}. \quad (3.4)$$

With this aim in mind, we firstly recall (see [18]) the superintegrability properties of the Hamiltonian $\hat{\mathcal{H}}$ (3.4) which are summarized in the following statement, and that are worth to be compared with Proposition 1.

Proposition 2. *Let $\hat{\mathcal{H}}$ be the quantum Hamiltonian given by (3.4). Then:*

(i) $\hat{\mathcal{H}}$ commutes with the $(2N-3)$ quantum angular momentum operators ($m = 2, \dots, N$):

$$\hat{C}^{(m)} = \sum_{1 \leq i < j \leq m} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \quad \hat{C}_{(m)} = \sum_{N-m < i < j \leq N} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \quad \hat{C}^{(N)} = \hat{C}_{(N)} = \hat{\mathbf{L}}^2, \quad (3.5)$$

where $\hat{\mathbf{L}}^2$ is the total quantum angular momentum operator, as well as with the N Runge–Lenz operators given by ($i = 1, \dots, N$):

$$\hat{\mathcal{R}}_i = \frac{1}{2} \sum_{j=1}^N \hat{p}_j (\hat{q}_j \hat{p}_i - \hat{q}_i \hat{p}_j) + \frac{1}{2} \sum_{j=1}^N (\hat{q}_j \hat{p}_i - \hat{q}_i \hat{p}_j) \hat{p}_j + \frac{\hat{q}_i}{|\hat{\mathbf{q}}|} \left(\eta \hat{\mathcal{H}} + k \right). \quad (3.6)$$

(ii) Each of the three sets $\{\hat{\mathcal{H}}, \hat{C}^{(m)}\}$, $\{\hat{\mathcal{H}}, \hat{C}_{(m)}\}$ ($m = 2, \dots, N$) and $\{\hat{\mathcal{R}}_i\}$ ($i = 1, \dots, N$) is formed by N algebraically independent commuting operators.

(iii) The set $\{\hat{\mathcal{H}}, \hat{C}^{(m)}, \hat{C}_{(m)}, \hat{\mathcal{R}}_i\}$ for $m = 2, \dots, N$ with a fixed index i is formed by $(2N - 1)$ algebraically independent operators.

(iv) $\hat{\mathcal{H}}$ is formally self-adjoint on the Hilbert space of square-integrable functions with respect to the scalar product

$$\langle \Psi | \Phi \rangle = \int_{\mathbb{R}^N} \overline{\Psi(\mathbf{q})} \Phi(\mathbf{q}) \left(1 + \frac{\eta}{|\mathbf{q}|} \right) d\mathbf{q}. \quad (3.7)$$

With this result at hand, the maximal superintegrability of the conformal Laplacian quantization $\hat{\mathcal{H}}_c$ can be obtained by making use of the similarity transformation (3.3) in order to get the quantum integrals for $\hat{\mathcal{H}}_c$ starting from the ones for $\hat{\mathcal{H}}$. More explicitly, we have:

Proposition 3. (i) The quantum Hamiltonian $\hat{\mathcal{H}}_c$ given by (3.2) commutes with the operators (3.5) as well as with the following N operators of Runge–Lenz type ($i = 1, \dots, N$):

$$\begin{aligned} \hat{\mathcal{R}}_{c,i} = & \frac{1}{2} \sum_{j=1}^N \left(\hat{p}_j + i\hbar \eta \frac{(N-2)\hat{q}_j}{4(\eta + |\hat{\mathbf{q}}|) \hat{\mathbf{q}}^2} \right) (\hat{q}_j \hat{p}_i - \hat{q}_i \hat{p}_j) \\ & + \frac{1}{2} \sum_{j=1}^N (\hat{q}_j \hat{p}_i - \hat{q}_i \hat{p}_j) \left(\hat{p}_j + i\hbar \eta \frac{(N-2)\hat{q}_j}{4(\eta + |\hat{\mathbf{q}}|) \hat{\mathbf{q}}^2} \right) + \frac{\hat{q}_i}{|\hat{\mathbf{q}}|} \left(\eta \hat{\mathcal{H}}_c + k \right). \end{aligned}$$

(ii) Each of the three sets $\{\hat{\mathcal{H}}_c, \hat{C}^{(m)}\}$, $\{\hat{\mathcal{H}}_c, \hat{C}_{(m)}\}$ ($m = 2, \dots, N$) and $\{\hat{\mathcal{R}}_{c,i}\}$ ($i = 1, \dots, N$) is formed by N algebraically independent commuting operators.

(iii) The set $\{\hat{\mathcal{H}}_c, \hat{C}^{(m)}, \hat{C}_{(m)}, \hat{\mathcal{R}}_{c,i}\}$ for $m = 2, \dots, N$ with a fixed index i is formed by $(2N - 1)$ algebraically independent operators.

(iv) $\hat{\mathcal{H}}_c$ is formally self-adjoint on the Hilbert space $L^2(\mathcal{M})$ with its natural scalar product

$$\langle \Psi | \Phi \rangle_c = \int_{\mathbb{R}^N} \overline{\Psi(\mathbf{q})} \Phi(\mathbf{q}) \left(1 + \frac{\eta}{|\mathbf{q}|} \right)^{N/2} d\mathbf{q}.$$

Notice that the multiplication operator e^f (3.3) also defines a unitary transformation mapping

$$L^2(\mathbb{R}^N, (1 + \eta/|\mathbf{q}|) d\mathbf{q}) \quad \text{into} \quad L^2(\mathbb{R}^N, (1 + \eta/|\mathbf{q}|)^{N/2} d\mathbf{q}),$$

which is the natural L^2 space defined by the Riemannian metric. Therefore, according to the above statement, $\hat{\mathcal{H}}_c$ can be seen as an appropriate quantization of the classical Hamiltonian (1.1), as it manifestly preserves the maximal superintegrability of the classical system.

3.2 Radial Schrödinger equation

By mimicking the same approach presented in [15], it is straightforward to prove that the quantum radial Hamiltonian corresponding to (3.4) is

$$\hat{\mathcal{H}} = \frac{r}{2(\eta + r)} \left(\frac{1}{r^{N-1}} \hat{p}_r r^{N-1} \hat{p}_r + \frac{\hat{\mathbf{L}}^2}{r^2} - \frac{2k}{r} \right), \quad (3.8)$$

where $\hat{\mathbf{L}}^2$ is the square of the total quantum angular momentum operator, given by

$$\hat{\mathbf{L}}^2 = \sum_{j=1}^{N-1} \left(\prod_{k=1}^{j-1} \frac{1}{\sin^2 \theta_k} \right) \frac{1}{(\sin \theta_j)^{N-1-j}} \hat{p}_{\theta_j} (\sin \theta_j)^{N-1-j} \hat{p}_{\theta_j}.$$

Here the hyperspherical coordinates (2.4) have been used together with

$$\hat{p}_r = -i\hbar \frac{\partial}{\partial r}, \quad \hat{p}_{\theta_j} = -i\hbar \frac{\partial}{\partial \theta_j}, \quad j = 1, \dots, N-1. \quad (3.9)$$

After reordering terms and by introducing the differential operators (3.9) within the Hamiltonian (3.8), we arrive at the following Schrödinger equation,

$$\frac{r}{2(\eta + r)} \left(-\hbar^2 \partial_r^2 - \frac{\hbar^2(N-1)}{r} \partial_r + \frac{\hat{\mathbf{L}}^2}{r^2} - \frac{2k}{r} \right) \Psi(r, \boldsymbol{\theta}) = E \Psi(r, \boldsymbol{\theta}), \quad (3.10)$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{N-1})$. By taking into account that the hyperspherical harmonics $Y(\boldsymbol{\theta})$ are such that

$$\hat{\mathbf{L}}^2 Y(\boldsymbol{\theta}) = \hat{C}_{(N)} Y(\boldsymbol{\theta}) = \hbar^2 l(l + N - 2) Y(\boldsymbol{\theta}), \quad l = 0, 1, 2, \dots,$$

where l is the angular momentum quantum number, the equation (3.10) admits a complete set of factorized solutions of the form

$$\Psi(r, \boldsymbol{\theta}) = \Phi(r) Y(\boldsymbol{\theta}),$$

and, moreover,

$$\hat{C}_{(m)} \Psi = c_m \Psi, \quad m = 2, \dots, N,$$

where the eigenvalues c_m of the operators $\hat{C}_{(m)}$ (3.5) are related to the $(N-1)$ quantum numbers of the angular observables in the form

$$c_k \leftrightarrow l_{k-1}, \quad k = 2, \dots, N-1, \quad c_N \leftrightarrow l.$$

Therefore, we can write

$$Y(\boldsymbol{\theta}) \equiv Y_{c_{N-1}, \dots, c_2}^{c_N}(\theta_1, \theta_2, \dots, \theta_{N-1}) \equiv Y_{l_{N-2}, \dots, l_1}^l(\theta_1, \theta_2, \dots, \theta_{N-1}),$$

and the radial Schrödinger equation provided by $\hat{\mathcal{H}}$ reads

$$\frac{r}{2(\eta + r)} \left(-\hbar^2 \left(\frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{l(l+N-2)}{r^2} \right) - \frac{2k}{r} \right) \Phi(r) = E \Phi(r). \quad (3.11)$$

Since the radial Hamiltonians $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}_c$ are related through the unitary transformation (3.3), namely,

$$\hat{\mathcal{H}}_c = \left(1 + \frac{\eta}{r}\right)^{(2-N)/4} \hat{\mathcal{H}} \left(1 + \frac{\eta}{r}\right)^{(N-2)/4},$$

the radial equation coming from the conformal Laplacian quantization (3.2) is found to be

$$\left\{ -\frac{\hbar^2 r}{2(\eta + r)} \left(\frac{d^2}{dr^2} + \left(\frac{N-1}{r} - \frac{\eta(N-2)}{2r(\eta + r)} \right) \frac{d}{dr} - \frac{l(l+N-2)}{r^2} \right) - \frac{k}{\eta + r} + \hbar^2 \eta(N-2) \frac{4(N-3)r + 3\eta(N-2)}{32r(\eta + r)^3} \right\} \Phi_c(r) = E \Phi_c(r). \quad (3.12)$$

Therefore, the two radial equations (3.11) and (3.12) will share the same energy spectrum and their radial wave functions will be related through

$$\Phi_c(r) = \left(1 + \frac{\eta}{r}\right)^{(2-N)/4} \Phi(r). \quad (3.13)$$

4 Spectrum and eigenfunctions

In this section we shall compute, in a rigorous manner, the (continuous and discrete) spectrum and eigenfunctions of the quantum Hamiltonian $\hat{\mathcal{H}}_c$ (3.2). Although we will focus on the case $\eta > 0$ and $k > 0$, which is particularly interesting due to its connection with the Coulomb problem, we will also comment on the remaining possibilities for the signs of η and k .

To begin with, let us observe that the spherical symmetry of $\hat{\mathcal{H}}_c$ (3.2) leads us to decompose

$$L^2(\mathcal{M}) = \bigoplus_{l \in \mathbb{N}} L^2(\mathbb{R}^+, d\nu) \otimes \mathcal{Y}_l,$$

where $d\nu = r^{N-1}(1 + \eta/r)^{N/2}dr$ and \mathcal{Y}_l is the finite-dimensional space of (generalized) spherical harmonics, defined by

$$\mathcal{Y}_l := \{Y \in L^2(\mathbb{S}^{N-1}) : \Delta_{\mathbb{S}^{N-1}} Y = -l(l+N-2)Y\},$$

where \mathbb{N} stands for the set of nonnegative integers and $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplacian on the $(N-1)$ D sphere \mathbb{S}^{N-1} . We recall that \mathcal{Y}_l consists of the restriction to the sphere of the harmonic polynomials in \mathbb{R}^N that are homogeneous of degree l .

This decomposition implies that the wave function can be written as

$$\Psi_c(\mathbf{q}) = \sum_{l \in \mathbb{N}} Y_l(\boldsymbol{\theta}) \Phi_{c,l}(r),$$

with $\boldsymbol{\theta} = \mathbf{q}/r \in \mathbb{S}^{N-1}$, $r = |\mathbf{q}|$ and $Y_l \in \mathcal{Y}_l$.

In view of the expression for the radial effective potential (see (4.2) below), it is not hard to see that $\hat{\mathcal{H}}_c$ can be regarded as a densely defined self-adjoint operator on $L^2(\mathcal{M})$ that, by virtue of the above decomposition, can be written as

$$\hat{\mathcal{H}}_c = \bigoplus_{l \in \mathbb{N}} \hat{H}_{c,l} \otimes \text{id}_{\mathcal{Y}_l}.$$

From the expression of the potentials we see that each operator $\hat{H}_{c,l}$,

$$\hat{H}_{c,l} = -\frac{\hbar^2 r}{2(\eta + r)} \left(\frac{d^2}{dr^2} + \left(\frac{N-1}{r} - \frac{\eta(N-2)}{2r(\eta+r)} \right) \frac{d}{dr} - \frac{l(l+N-2)}{r^2} \right) - \frac{k}{\eta+r} + \hbar^2 \eta(N-2) \frac{4(N-3)r + 3\eta(N-2)}{32r(\eta+r)^3}, \quad (4.1)$$

can be taken to be the Friedrichs extension of the above differential operator acting on the space of smooth, compactly supported functions $C_0^\infty(\mathbb{R}^+)$ (see e.g. (3.12)). Therefore, the spectrum of $\hat{\mathcal{H}}_c$ is just

$$\text{spec}(\hat{\mathcal{H}}_c) = \overline{\bigcup_{l \in \mathbb{N}} \text{spec}(\hat{H}_{c,l})}.$$

In order to analyze the spectrum, the quantum effective potential $\hat{\mathcal{U}}_{\text{eff},l}$ will be helpful. This can be obtained by applying the change of radial variable $Q = Q(r)$ given by (2.10) together with a transformation of the radial wave function $\Phi_{c,l}(r) \mapsto \varphi(Q(r))$ and by imposing that these transformations map the Schrödinger equation $\hat{H}_{c,l}\Phi_{c,l} = E\Phi_{c,l}$ into

$$\left(-\frac{\hbar^2}{2} \frac{d^2}{dQ^2} + \hat{\mathcal{U}}_{\text{eff},l}(Q) \right) \varphi(Q) = E \varphi(Q).$$

This requires to introduce

$$\Phi_{c,l}(r) = \frac{r^{(1-N)/2}}{(1 + \frac{\eta}{r})^{(N-1)/4}} \varphi(Q)$$

in the radial Schrödinger equation provided by the Hamiltonian (4.1), thus yielding

$$\hat{\mathcal{U}}_{\text{eff},l}(r) = \frac{r}{2(\eta+r)} \left(-\frac{\hbar^2(\eta^2 + 4r^2)}{16r^2(\eta+r)^2} + \frac{\hbar^2}{r^2} \left(l(l+N-2) + \frac{(N-2)^2}{4} \right) - \frac{2k}{r} \right). \quad (4.2)$$

The behavior of the resulting quantum effective potential deserves some comments since, in contrast to the classical case, $\hat{\mathcal{U}}_{\text{eff},l}(r)$ depends on the dimension N of the Taub-NUT manifold:

- For $N \geq 3$, the behavior of the effective potential is always similar to that of the classical one (2.11) (see figure 5), irrespectively of the value of l . In particular, even in the case that $l = 0$, the quantum effective potential is such that

$$\lim_{r \rightarrow 0} \hat{\mathcal{U}}_{\text{eff},l}(r) = +\infty, \quad \lim_{r \rightarrow \infty} \hat{\mathcal{U}}_{\text{eff},l}(r) = 0. \quad (4.3)$$

Moreover, $\hat{\mathcal{U}}_{\text{eff},l}$ has always a unique minimum at r_{\min} whose $\eta \rightarrow 0$ non-deformed Coulomb limit is given by

$$\eta = 0 : \quad r_{0,\min} = \frac{\hbar^2}{k} (l(l+N-2) + (N-1)(N-3)/4),$$

$$\hat{\mathcal{U}}_{\text{eff},l}(r_{0,\min}) = -\frac{2k^2}{\hbar^2(4l(l+N-2) + (N-1)(N-3))},$$

with the exception of the non-deformed case with $N = 3$ and $l = 0$ which gives $r_{0,\min} = 0$ and $\hat{\mathcal{U}}_{\text{eff},l}(r_{0,\min}) \rightarrow -\infty$.

- When $N = 2$, the effective potential (4.2) reduces to

$$\hat{\mathcal{U}}_{\text{eff},l}(r) = \frac{r}{2(\eta + r)} \left(-\frac{\hbar^2(\eta^2 + 4r^2)}{16r^2(\eta + r)^2} + \frac{\hbar^2 l^2}{r^2} - \frac{2k}{r} \right).$$

If $l = 0$ we obtain that

$$\lim_{r \rightarrow 0} \hat{\mathcal{U}}_{\text{eff},0}(r) = -\infty, \quad \lim_{r \rightarrow \infty} \hat{\mathcal{U}}_{\text{eff},0}(r) = 0,$$

so that $\hat{\mathcal{U}}_{\text{eff},0}$ has no local minima and always takes negative values. On the contrary, for $l \geq 1$ we find that $\hat{\mathcal{U}}_{\text{eff},l}$ has the same limiting values (4.3) and has only one local minimum, where the effective potential is negative.

4.1 The case $k > 0$, $\eta > 0$

Let us now compute the discrete eigenvalues and eigenfunctions of $\hat{\mathcal{H}}_c$ when k and η are positive. To begin with, let us recall that, with $k > 0$, the radial part $\psi_{n,l}(r)$ of the eigenfunctions for the standard ND Coulomb problem satisfies the following Schrödinger equation corresponding to a spherical harmonic of degree l :

$$\left(-\hbar^2 \left(\frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} \right) + \frac{\hbar^2 l(l+N-2)}{r^2} - \frac{2k}{r} \right) \psi_{n,l}(r) = 2E_{n,l}^0 \psi_{n,l}(r), \quad (4.4)$$

with eigenvalues

$$E_{n,l}^0 = -\frac{k^2}{2\hbar^2 \left(n + l + \frac{N-1}{2} \right)^2}. \quad (4.5)$$

The explicit expression of $\psi_{n,l}$ is given, in terms of generalized Laguerre polynomials, by

$$\psi_{n,l}(r) = r^l \exp \left(-\frac{kr}{\hbar^2 \left(n + l + \frac{N-1}{2} \right)} \right) L_n^{2l+N-2} \left(\frac{2kr}{\hbar^2 \left(n + l + \frac{N-1}{2} \right)} \right), \quad (4.6)$$

up to the corresponding normalization constant.

Now, the solution for the eigenvalue problem of the deformed Coulomb Hamiltonian $\hat{\mathcal{H}}$ can be obtained if we realize that (3.11) can be rewritten in the form

$$\left(-\hbar^2 \left(\frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{l(l+N-2)}{r^2} \right) - \frac{2k}{r} \right) \Phi(r) = 2E \left(1 + \frac{\eta}{r} \right) \Phi(r), \quad (4.7)$$

which is nothing but the equation (4.4)

$$\left(-\hbar^2 \left(\frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} \right) + \frac{\hbar^2 l(l+N-2)}{r^2} - \frac{2K}{r} \right) \Phi(r) = 2E \Phi(r),$$

provided that we set

$$K = k + \eta E. \quad (4.8)$$

This means that the eigenvalue problem for the Hamiltonian $\hat{\mathcal{H}}$ is formally the standard Coulomb problem with a new coupling constant K that depends on the initial coupling

constant k , the energy E and the η parameter. This fact does not immediately yield the eigenvalues of the problem, however, because the integrability (and boundary) conditions that one must impose are different in this case. Note that this is analogous to what happens with the quantum harmonic oscillator eigenvalue problem with respect to the superintegrable oscillator defined on the Darboux III curved space [15]. Consequently, we shall directly solve the equation (4.7) and next analyze the corresponding results.

For $E < 0$, it can be shown that the general solution of (4.7) must be of the form

$$\Phi(\rho) = \rho^l e^{-\rho/2} (c_1 U_{-\alpha}^\beta(\rho) + c_2 M_{-\alpha}^\beta(\rho)), \quad (4.9)$$

where $U_{-\alpha}^\beta(\rho)$ and $M_{-\alpha}^\beta(\rho)$ stand for two independent confluent hypergeometric functions (called, respectively, Tricomi and Kummer functions) and we have set

$$\rho = \frac{2r\sqrt{-2E}}{\hbar}, \quad \alpha = \frac{k + \eta E}{\hbar\sqrt{-2E}} - l - \frac{N-1}{2}, \quad \beta = 2l + N - 1. \quad (4.10)$$

Let us recall that the large- ρ asymptotic behavior of the above functions is

$$U_{-\alpha}^\beta(\rho) \sim \rho^\alpha [1 + o(1)], \quad M_{-\alpha}^\beta(\rho) \sim e^\rho \rho^{-\alpha-\beta} \left[1 + o(1) \right],$$

where $o(1)$ stands for a quantity that tends to zero as $\rho \rightarrow \infty$ and Γ is the Gamma function. Therefore, we infer that the eigenfunction (4.9) will not be square-integrable at infinity, with respect to the corresponding radial measure (3.7), unless α is a nonnegative integer n , so that the hypergeometric series appearing in the definition of $M_{-\alpha}^\beta(\rho)$ collapses to a finite sum, yielding the generalized Laguerre polynomials $L_n^{\beta-1}(\rho)$ of degree $\alpha = n$. Moreover, while $M_{-\alpha}^\beta(\rho)$ remains bounded at 0, $U_{-\alpha}^\beta(\rho)$ diverges badly at the origin, so we must take $c_1 = 0$.

Hence, by substituting $\alpha = n$ in (4.10) we obtain the quadratic equation

$$\eta^2 E^2 + 2 \left(k\eta + \hbar^2 \left(n + l + \frac{N-1}{2} \right)^2 \right) E + k^2 = 0, \quad (4.11)$$

which we can readily solve to get the following explicit expression for the discrete eigenvalues of $\hat{\mathcal{H}}_c$:

$$E_{n,l} = \frac{-k^2}{\hbar^2 \left(n + l + \frac{N-1}{2} \right)^2 + k\eta + \sqrt{\hbar^4 \left(n + l + \frac{N-1}{2} \right)^4 + 2\hbar^2 k\eta \left(n + l + \frac{N-1}{2} \right)^2}}. \quad (4.12)$$

We remark that the positive sign in the square root is chosen in such a manner that the limit $\eta \rightarrow 0$ exists and returns the non-deformed discrete spectrum (4.5) and, moreover, this result ensures that the new coupling constant K (4.8) remains positive for all values of n, l . In this respect, notice also that the equation (4.11) can be rewritten in terms of K in the form

$$E = -\frac{K^2}{2\hbar^2 \left(n + l + \frac{N-1}{2} \right)^2}, \quad (4.13)$$

to be compared with (4.5).

Then the eigenstates $\Phi(r)$ for $\hat{\mathcal{H}}$ can straightforwardly be obtained, in terms of K , by introducing ρ and β given in (4.10), $\alpha = n$ and the relation (4.13) in the eigenfunction (4.9) with $c_1 = 0$. Next, the similarity transformation (3.13) yields the eigenstates for $\hat{\mathcal{H}}_c$; namely

$$\Phi_c(r) = \left(1 + \frac{\eta}{r}\right)^{\frac{2-N}{4}} r^l \exp\left(-\frac{Kr}{\hbar^2 \left(n + l + \frac{N-1}{2}\right)}\right) L_n^{2l+N-2}\left(\frac{2Kr}{\hbar^2 \left(n + l + \frac{N-1}{2}\right)}\right), \quad (4.14)$$

where one must keep in mind that, in fact, K depends on η and $E_{n,l}$ (4.12).

In view of the asymptotic behavior of the effective potential, standard results for one-dimensional differential operators show [26, Theorem XIII.7.66] that the continuous spectrum for $\hat{\mathcal{H}}_c$ is the positive real line and that there are no eigenvalues embedded in the continuum. Hence we can now summarize all the above results as follows.

Theorem 1. *Let $\hat{\mathcal{H}}_c$ be the quantum Hamiltonian (3.2) with $k > 0$ and $\eta > 0$. Then:*

- (i) *The continuous spectrum of $\hat{\mathcal{H}}_c$ is given by $[0, \infty)$. Moreover, there are no embedded eigenvalues and the singular spectrum is empty.*
- (ii) *$\hat{\mathcal{H}}_c$ has an infinite number of eigenvalues $E_{n,l}$, depending only on the sum $(n + l)$ and accumulating at 0.*
- (iii) *The eigenvalues of $\hat{\mathcal{H}}_c$ are of the form (4.12) and $\Psi_c = \Phi_c(r)Y(\boldsymbol{\theta})$, determined by (4.14), is eigenfunction of $\hat{\mathcal{H}}_c$ with eigenvalue $E_{n,l}$.*

Notice that, in particular, the bound states of this system satisfy

$$\lim_{n,l \rightarrow \infty} E_{n,l} = 0, \quad \lim_{n \rightarrow \infty} (E_{n+1} - E_n) = 0, \quad n = n + l.$$

As it can be appreciated in Figure 6, deviations from the spectrum of the quantum Coulomb problem are significant for the low energy states, since the effect of the deformation is essentially a shift $r \rightarrow r + \eta$ in the standard Coulomb potential. In fact, in the limit $\eta \rightarrow 0$ of (4.12) the well-known formula for the standard Coulomb eigenvalues $E_{n,l}^0$ (4.5) is recovered. The deformation on the spectrum can be better appreciated through its power series expansion in η given by

$$E_{n,l} = E_{n,l}^0 + \eta \frac{k^3}{2\hbar^4 \left(n + l + \frac{N-1}{2}\right)^4} - \eta^2 \frac{5k^4}{8\hbar^6 \left(n + l + \frac{N-1}{2}\right)^6} + O(\eta^3).$$

4.2 A formal Lie-algebraic derivation of the spectrum

Here we will show how the symmetry Lie algebra generated by the constants of motion strongly suggests that the eigenvalues of the system (for $k > 0$ and $\eta > 0$) are indeed of the form (4.12) which we rigorously have just found. In this subsection we will present a formal algebraic derivation of the spectrum, although the reader is advised to take it with a pinch of salt, since subtle technical issues related to the domain of the operators make this derivation non-rigorous; indeed, we shall see in the following subsection that, for this reason, the case of negative k and/or η does not fit into this algebraic framework.

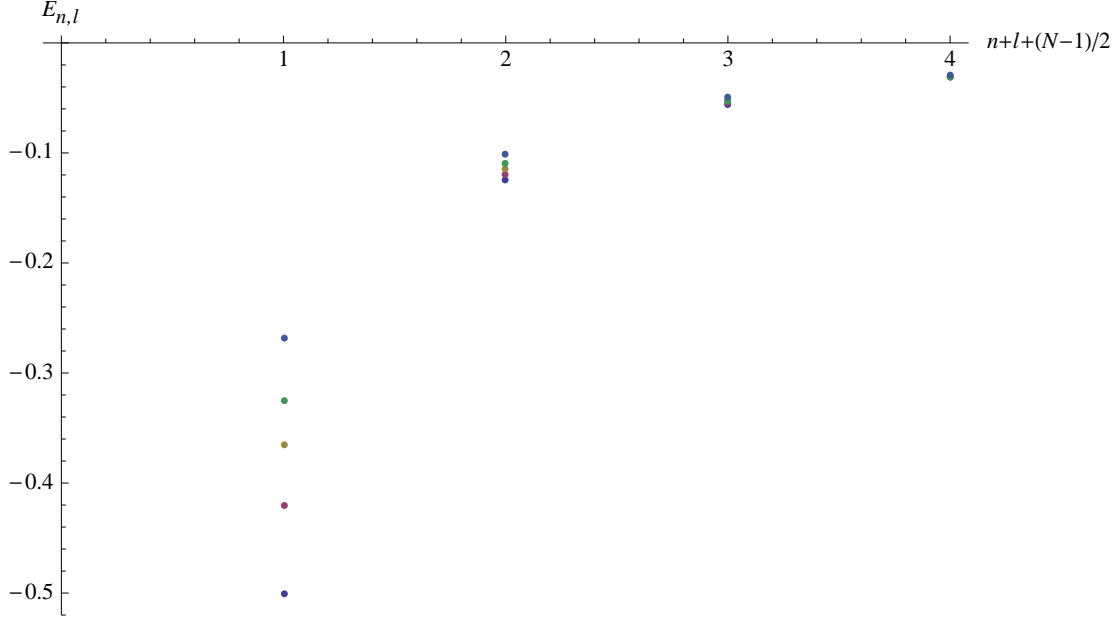


Figure 6: Discrete spectrum (4.12) for the fundamental and the three first excited states of the Hamiltonian $\hat{\mathcal{H}}_c$ when $\eta = \{0, 0.2, 0.4, 0.6, 1\}$ with $\hbar = k = 1$ and $N \geq 3$. Note that the effect of the η deformation is quite strong for the fundamental state, since it comes from the shift $r \rightarrow r + \eta$ in the usual Coulomb potential.

Let us start off with the sum of the squared Runge–Lenz operators introduced in (3.6), which turns out to be

$$\hat{\mathbf{R}}^2 = \sum_{i=1}^N \hat{\mathcal{R}}_i^2 = 2\hat{\mathcal{H}} \left(\hat{\mathbf{L}}^2 + \hbar^2 \frac{(N-1)^2}{4} \right) + \left(\eta \hat{\mathcal{H}} + k \right)^2, \quad (4.15)$$

which is the quantum counterpart of the relation (2.2). Now, if similarly to the classical case (2.3), we introduce the following operators (that can be only defined on the set of eigenfunctions of \hat{H}_η)

$$\hat{J}_{0i} = \frac{\hat{\mathcal{R}}_i}{\sqrt{-2\hat{\mathcal{H}}}},$$

then the equation (4.15) becomes formally

$$\sum_{i=1}^N \hat{J}_{0i}^2 + \hat{\mathbf{L}}^2 + \frac{\hbar^2(N-1)^2}{4} = -\frac{(\eta \hat{\mathcal{H}} + k)^2}{2\hat{\mathcal{H}}}, \quad (4.16)$$

where $\sum_{i=1}^N \hat{J}_{0i}^2 + \hat{\mathbf{L}}^2$ is the quadratic Casimir operator of the $\mathfrak{so}(N+1)$ Lie algebra. Therefore, if this operator has its usual domain, its eigenvalues are

$$\hbar^2 \mathbf{n}(\mathbf{n} + (N+1) - 2) = \hbar^2 \mathbf{n}(\mathbf{n} + N - 1).$$

This implies that the eigenvalues of the l.h.s. of equation (4.16) should be $\hbar^2(\mathbf{n} + \frac{N-1}{2})^2$. Now, if we act with (4.16) onto an eigenfunction of $\hat{\mathcal{H}}$ with eigenvalue E , we formally get

$$-\frac{(\eta E + k)^2}{2E} = \hbar^2 \left(\mathbf{n} + \frac{N-1}{2} \right)^2,$$

which is just the equation (4.11) (and also (4.13)) for the energies where $\mathbf{n} = n + l$.

4.3 The remaining cases

For the sake of completeness, let us qualitatively describe the spectrum of the system for the remaining possible values of η and k .

The case $\eta > 0, k < 0$

In this case the manifold is again $\mathcal{M} = \mathbb{R}^N \setminus \{\mathbf{0}\}$, but the potential is repulsive. The Hamiltonian $\hat{H}_{c,l}$ can be again defined as the Friedrichs extension of the action of the differential operator on $C_0^\infty(\mathbb{R}^+)$. In view of our formulas for the quantum effective potential on the space of degree- l spherical harmonics, an easy application of [26, Theorem XIII.7.66] shows that the spectrum of $\hat{H}_{c,l}$ is purely absolutely continuous and given by $[0, \infty)$.

The case $\eta < 0, k > 0$

Now the manifold is $\mathcal{M} = \{\mathbf{q} \in \mathbb{R}^N : |\mathbf{q}| > |\eta|\}$ and the potential is attractive, so one expects to have eigenvalues. The Hamiltonian $\hat{H}_{c,l}$ is defined as the Friedrichs extension of the operator with domain $C_0^\infty((|\eta|, \infty))$; its continuous spectrum is $[0, \infty)$ and again there is no embedded eigenvalues or singular continuous spectrum.

Therefore, the eigenvalues we are looking for would be those values of E for which the function Φ (4.9) is square-integrable at infinity, with respect to the induced radial measure, and such that this satisfies the boundary condition $\Phi(|\eta|) = 0$. In principle, one has the values of $E < 0$ for which the confluent hypergeometric function U satisfies the boundary condition

$$U_{-\alpha}^{2l+N-1} \left(\frac{2|\eta|\sqrt{-2E}}{\hbar} \right) = 0,$$

where α also depends on E via (4.10). These eigenfunctions correspond to set $c_2 = 0$ in (4.9). This kind of eigenvalues cannot be computed in closed form, but it is not hard to see that one can take specific values of k and η where there are indeed eigenvalues that satisfy these conditions.

However, we remark that to obtain in a rigorous manner the eigenvalues for Φ with $c_2 \neq 0$, that is, of the type $E_{n,l}$ (4.12), is not straightforward at all and a deeper analysis of this delicate case is deserved.

The case $\eta < 0, k < 0$

Now the manifold is $\mathcal{M} = \{\mathbf{q} \in \mathbb{R}^N : |\mathbf{q}| > |\eta|\}$ and the potential is repulsive. The Hamiltonian $\hat{H}_{c,l}$ is defined as the Friedrichs extension of the operator with domain $C_0^\infty((|\eta|, \infty))$ and the spectrum, which consists of the positive real line, is purely absolutely continuous.

5 Concluding remarks

In this paper we have presented a new exactly solvable quantum system in N dimensions, that has been obtained as the maximally superintegrable quantization of the Hamiltonian (1.1). Such quantization has been performed by making use of the conformal Laplacian prescription and its equivalence with the so-called “direct Schrödinger” one. It is worth mentioning that, as it was shown in [15], both quantization approaches can also be related to the *position-dependent mass* quantization (see, for instance [27, 28, 29])

$$\hat{\mathcal{H}}_{\text{PDM}} = -\frac{\hbar^2}{2} \nabla \cdot \frac{1}{f(r)^2} \nabla + \mathcal{U}(\mathbf{q}),$$

where $f(r)$ is the conformal factor of the metric (1.4).

It is also important to remark that, although the potential of the system (1.1), namely

$$\mathcal{U}(\mathbf{q}) = -\frac{k}{\eta + |\mathbf{q}|}$$

can be interpreted as an η -deformation of the Coulomb problem on the ND Euclidean space, \mathcal{U} is by no means superintegrable (and, therefore, exactly solvable) on such flat space. It turns out that in order to recover a superintegrable system containing this deformed Coulomb potential, the kinetic energy term has also to be η -deformed, and the outcome of this deformation is just a metric associated with the Taub–NUT space whose curvature (1.3) is again controlled by the deformation parameter η . In this sense, the Hamiltonian (1.1) is a quite singular system in which the *same* parameter η plays both a dynamical role (in the potential term) and a geometric one (in the kinetic energy), and both roles have to be exactly tuned in order to allow superintegrability to arise.

Therefore, the system (1.1) and its superintegrable conformal Laplacian quantization $\hat{\mathcal{H}}_c$ (3.2) can be indeed considered a deformation (in the two abovementioned dynamical and geometric senses) of the Euclidean Coulomb system, and such deformed quantum system can be fully solved by making use of the very same techniques used in the former. Nevertheless, we recall that the system (1.1) cannot be properly called the Coulomb problem on the Taub–NUT space with metric (1.4). In fact, the geometrical definitions of the intrinsic Coulomb \mathcal{U}_C and oscillator \mathcal{U}_O potentials on a spherically symmetric space are, respectively, given by (see [24] for details)

$$\mathcal{U}_C(r) := \int^r \frac{dr'}{r'^2 f(r')}, \quad \mathcal{U}_O(r) := \frac{1}{\mathcal{U}_C(r)^2},$$

up to multiplicative and additive constants, where $f(r)$ is the conformal factor of the metric (1.4). Now, if we apply these expressions to the Taub–NUT space whose conformal factor is (1.5) we find that, whenever $\eta \neq 0$, the potential \mathcal{U} in (1.1) defines an *intrinsic oscillator* potential \mathcal{U}_O on the curved space \mathcal{M} with metric (1.2) that would be given by (see [18] for more details)

$$\mathcal{U}_O = C \frac{r}{\eta + r} + D,$$

where C and D are real constants. Hence, if we set $C = k/\eta$ and $D = -C$, we get that $\mathcal{U}_O \equiv \mathcal{U}$. From a physical viewpoint, this can be understood in the sense that the potential \mathcal{U} takes a finite value on $r = 0$ (see Figure 4), which means that the characteristic Coulomb potential singularity at the origin has been suppressed by the deformation.

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